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On the support of tempered distributions

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Abstract. We show that, given a tempered distribution S whose Fourier transform is a function of polynomial growth, a point x in $\mathbb{R}^n$ is outside the support of S if and only if the Fourier integral of S is summable in Bochner-Riesz means to zero uniformly on a neighbourhood of x.

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1. Introduction. In [4, pp. 54–55] Kahane and Salem and in [6] Walter proved results linking the support of a periodic distribution and its Fourier series which, transposed on $\mathbb{R}$, can be stated as follows. If S is a tempered distribution with $\mathcal{F}S \in C_0(\mathbb{R})$, then $x_0 \in \mathbb{R}$ is outside the support of S if and only if $\lim_{N\to+\infty} \int_{-N}^{N} \mathcal{F}S(t)\mathrm{e}^{2\pi ixt}dt = 0$ for all x in a neighbourhood of x_0. If T is a compactly supported distribution then $x \notin \operatorname{supp} T$ implies

$$\lim_{N\to+\infty} \int_{-N}^{N} (1 - |t|/N)^k \mathcal{F}T(t)\mathrm{e}^{2\pi ixt}\, dt = 0 \tag{1.1}$$

for some $k \geq 0$; moreover the reciprocal is false: (1.1) holds at every $x \in \mathbb{R}$ for $T = \delta_0'$ and $k = 2$.

The key to get a characterization of the support of T is to observe that (1.1) in fact holds *uniformly* on a neighbourhood of $x \notin \operatorname{supp} T$. The necessity and sufficiency of this condition was obtained by the first author for compactly supported distributions on all Euclidean spaces [2]; recently the second author has shown on $\mathbb{R}$ that this condition is necessary without restriction on the support of $T \in \mathcal{S}'(\mathbb{R})$ but with the assumption that $\mathcal{F}T$ be of polynomial growth [3].

Here we extend the reasonings of [3] to all euclidean spaces and thus show that, given $S \in \mathcal{S}'(\mathbb{R}^n)$ with $\mathcal{F}S$ a function of polynomial growth, a point x in $\mathbb{R}^n$ is outside the support of S if and only if the Fourier integral of S is summable in Bochner-Riesz means to 0 uniformly on a neighbourhood of x.

To prove this in section 4, we need an auxiliary lemma we state and prove in section 3. Section 2 introduces useful notations.

2. Preliminaries. We put $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Let $\lambda \geq 0$. We write $A_\lambda(\mathbb{R}^n)$ the set of functions φ on $\mathbb{R}^n$ $(n \geq 2)$ which are the Fourier transform

$$\varphi(y) = \mathcal{F}f(y) := \int\limits_{\mathbb{R}^n} f(t) \mathrm{e}^{-2\pi i (t|y)}\, dt$$

of an integrable function f on $\mathbb{R}^n$ such that $(1 + \|t\|)^\lambda f(t)$ is also integrable on $\mathbb{R}^n$. Here

$$(t|y)$$

denotes the scalar product of vectors $t, y \in \mathbb{R}^n$. We define a norm $\|\ \|_{A_\lambda}$ on $A_\lambda(\mathbb{R}^n)$ by

$$\|\mathcal{F}f\|_{A_\lambda} := \int\limits_{\mathbb{R}^n} (1 + \|t\|)^\lambda |f(t)|\, dt.$$

Note that $A_\lambda(\mathbb{R}^n) \subset A_0(\mathbb{R}^n)$, which is the Fourier Algebra of $\mathbb{R}^n$, and that $\mathcal{S}(\mathbb{R}^n)$ (the set of rapidly decreasing functions on $\mathbb{R}^n$) is dense in $A_\lambda(\mathbb{R}^n)$.

We write $\mathcal{S}'_\lambda(\mathbb{R}^n)$ the set of tempered distributions S on $\mathbb{R}^n$ whose Fourier transform $\mathcal{F}S$ is a function in L^∞_{loc} such that $\mathcal{F}S(t)(1 + \|t\|)^{-\lambda}$ is essentially bounded on $\mathbb{R}^n$. We define a norm on $\mathcal{S}'_\lambda(\mathbb{R}^n)$ by

$$\|S\|_{\mathcal{S}'_\lambda} := \text{ess sup}_{t \in \mathbb{R}^n} |\mathcal{F}S(t)|(1 + \|t\|)^{-\lambda}.$$

Note that $\mathcal{S}'_\lambda(\mathbb{R}^n) \supset \mathcal{S}'_0(\mathbb{R}^n)$, which is the set of pseudomeasures on $\mathbb{R}^n$, and that every distribution on $\mathbb{R}^n$ with compact support is in some $\mathcal{S}'_\lambda(\mathbb{R}^n)$.

We define a pairing $\langle\ ,\ \rangle_\lambda$ between $\mathcal{S}'_\lambda(\mathbb{R}^n)$ and $A_\lambda(\mathbb{R}^n)$ by

$$\begin{aligned} \langle S, \mathcal{F}f\rangle_\lambda &:= \int\limits_{\mathbb{R}^n} \mathcal{F}S(t) f(t)\, dt \\ &= \int\limits_{\mathbb{R}^n} \mathcal{F}S(t)(1 + \|t\|)^{-\lambda}(1 + \|t\|)^\lambda f(t)\, dt. \end{aligned}$$

We have $|\langle S, \varphi\rangle_\lambda| \leq \|S\|_{\mathcal{S}'_\lambda} \cdot \|\varphi\|_{A_\lambda}$; moreover, when $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle S, \varphi\rangle_\lambda = S(\varphi)$.

For $k \geq 0$, $N > 0$, $m, n \in \mathbb{N}$ and $y \in \mathbb{R}^m$, we put

$$(2.1) \qquad {}_kL^n_N(y) := \frac{\Gamma(k+1)}{\pi^k}\, N^{n/2-k}\, \frac{1}{\|y\|^{n/2+k}}\, J_{n/2+k}(2\pi N \|y\|)$$

where J_μ is the Bessel function of the first kind and order μ. When $m = n$ we have

$$ {}_kL_N^n(y) = \mathcal{F}\{(1 - \|t\|^2/N^2)_+^k\}(y) \tag{2.2} $$

by [5, theorem 4.15 p.171], where $f_+ := \max(f, 0)$ if f is a real valued function and the Fourier transform is taken over $\mathbb{R}^n$. Because of the radial nature of ${}_kL_N^n$, it makes sense to have the two formulas for it, though of course our definition is motivated by the Fourier transform formula (2.2).

Let us differentiate ${}_kL_N^n$. Since $\frac{d}{dz}\big(z^{-\mu}J_\mu(z)\big) = -z^{-\mu}J_{\mu+1}(z)$, we have

$$ \frac{\partial}{\partial y_j}\, {}_kL_N^n(y) = -2\pi y_j \cdot {}_kL_N^{n+2}(y) $$

for $y \in \mathbb{R}^n$ and $j = 1, \ldots, n$. By iteration we get for every multiindex $\alpha \neq 0$

$$ D^\alpha{}_kL_N^n(y) = \sum_{j=1}^{|\alpha|} P_j(y) \cdot {}_kL_N^{n+2j}(y), $$

where each P_j is a polynomial of degree $\leq j$. Since, for $\mu > -1$, $J_\mu(z) = O(1/\sqrt{z})$ as $z \to +\infty$ with $z \in \mathbb{R}$ [5, (3.12) p.158], we see that, if $k > (n-1)/2$, then $D^\alpha{}_kL_N^n \in L^1(\mathbb{R}^n)$ for all multi-indices $\alpha \in \mathbb{N}_0^n$ and that, if $k > (n-1)/2 + |\alpha|$, then, given $\delta > 0$ arbitrary,

$$ \lim_{N\to+\infty} \int\limits_{\|y\|\geq\delta} |D^\alpha{}_kL_N^n(y)|\, dy = 0. \tag{2.3} $$

Moreover, $D^\alpha{}_kL_N^n \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ and $k \geq 0$ and, if $k \geq (n-1)/2 + |\alpha|$, then, given $\delta > 0$ arbitrary, there exists a constant $K > 0$ depending only on n, k, $|\alpha|$ and δ such that, for all $N \geq 1$,

$$ \int\limits_{\|y\|\geq\delta} |D^\alpha{}_kL_N^n(y)|^2\, dy \leq K. \tag{2.4} $$

3. An auxiliary lemma.

Lemma. *Let $\psi \in L^2(\mathbb{R}^n)$ and $j_0 \in \mathbb{N}_0$. Assume that $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ satisfies $|f(x) - f(y)| \leq \psi(x)\|x - y\|$ for all x, $y \in \mathbb{R}^n$ with $\|x - y\| \leq \pi^{-1}\sqrt{(n+2)/5}\cdot 2^{-j_0}$. Then there exists a constant $C > 0$ depending only on n such that, for all $j \in \mathbb{N}_0$ with $j \geq j_0$,*

$$ \int\limits_{2^j\leq\|t\|\leq 2^{j+1}} |\mathcal{F}f(t)|\, dt \leq C \cdot 2^{j(n-2)/2} \cdot \|\psi\|_2. $$

Proof. We take $0 < r \leq \pi^{-1}\sqrt{(n+2)/5} \cdot 2^{-j_0}$ and put, for all $x \in \mathbb{R}^n$,

$$ g(x) := \frac{1}{\omega_n r^{n-1}} \int\limits_{S(0,r)} [f(x+y) - f(x)]\, d\sigma_r(y), $$

where $d\sigma_r$ is the rotation-invariant area element on the sphere $S(0,r)$ such that $\int_{S(0,r)} d\sigma_r(y) = \omega_n r^{n-1}$ with $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$. Hence g is integrable on $\mathbb{R}^n$ and

$$\begin{aligned}
\mathcal{F}g(t) &= \int\limits_{\mathbb{R}^n} \left(\frac{1}{\omega_n r^{n-1}} \int\limits_{S(0,r)} [f(x+y) - f(x)]\, d\sigma_r(y) \right) \mathrm{e}^{-2\pi i(x|t)} dx \\
&= \frac{1}{\omega_n r^{n-1}} \int\limits_{S(0,r)} \left(\int\limits_{\mathbb{R}^n} [f(x+y) - f(x)]\, \mathrm{e}^{-2\pi i(x|t)} dx \right) d\sigma_r(y) \\
&= \frac{1}{\omega_n r^{n-1}} \int\limits_{S(0,r)} [\mathcal{F}f(t)\mathrm{e}^{2\pi i(y|t)} - \mathcal{F}f(t)]\, d\sigma_r(y) \\
&= \mathcal{F}f(t) \left(\frac{1}{\omega_n r^{n-1}} \int\limits_{S(0,r)} \mathrm{e}^{2\pi i(y|t)} d\sigma_r(y) - 1 \right).
\end{aligned}$$

But

$$\begin{aligned}
\int\limits_{S(0,r)} \mathrm{e}^{2\pi i(y|t)} d\sigma_r(y) &= \int\limits_{S(0,1)} \mathrm{e}^{2\pi i(ru|t)} r^{n-1}\, d\sigma_1(u) \\
&= r^{n-1} \int\limits_{S(0,1)} \mathrm{e}^{2\pi i(u|rt)}\, d\sigma_1(u) \\
&= r^{n-1} 2\pi \|rt\|^{-\nu} J_\nu(2\pi r\|t\|)
\end{aligned}$$

where $\nu := (n-2)/2$ [5, p. 154]. Therefore

$$\mathcal{F}g(t) = \mathcal{F}f(t) \left(\frac{2\pi}{\omega_n r^\nu \|t\|^\nu}\, J_\nu(2\pi r\|t\|) - 1 \right).$$

Now, on the one hand we have, from the choice of r,

$$\begin{aligned}
|g(x)| &= \left| \frac{1}{\omega_n r^{n-1}} \int\limits_{S(0,r)} [f(x+y) - f(x)]\, d\sigma_r(y) \right| \\
&\leq \frac{1}{\omega_n r^{n-1}} \int\limits_{S(0,r)} |f(x+y) - f(x)|\, d\sigma_r(y) \\
&\leq \frac{1}{\omega_n r^{n-1}} \int\limits_{S(0,r)} \psi(x)\, \|y\|\, d\sigma_r(y) \\
&= \psi(x) r
\end{aligned}$$

for all $x \in \mathbb{R}$; hence $g \in L^2(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} |g(x)|^2 \, dx \le r^2 \int_{\mathbb{R}^n} \psi^2(x) \, dx.$$

On the other hand we find, by Plancherel,

$$\begin{aligned} \int_{\mathbb{R}^n} |g(x)|^2 \, dx &= \int_{\mathbb{R}^n} |\mathcal{F}g(t)|^2 \, dt \\ &= \int_{\mathbb{R}^n} |\mathcal{F}f(t)|^2 \cdot \left| \frac{2\pi}{\omega_n r^\nu \|t\|^\nu} \, J_\nu(2\pi r\|t\|) - 1 \right|^2 dt. \end{aligned}$$

But, according to [1, p.199], since $\nu \in \frac{1}{2}\mathbb{N}_0$ we have, for all $m \in \mathbb{N}_0$ and all $u \ge 0$,

$$J_\nu(u) \le \sum_{l=0}^{2m} \frac{(-1)^l}{l! \, \Gamma(1 + l + \nu)} \left(\frac{u}{2}\right)^{2l+\nu}.$$

Thus

$$1 - \frac{2\pi}{\omega_n r^\nu \|t\|^\nu} \, J_\nu(2\pi r\|t\|) \ge \frac{(\pi r\|t\|)^2}{\nu + 1} - \frac{(\pi r\|t\|)^4}{2(\nu + 2)(\nu + 1)}$$

for all $t \in \mathbb{R}^n$. Let $Q : \mathbb{R} \to \mathbb{R}$ be defined by

$$Q(u) := \frac{u^2}{\nu + 1} - \frac{u^4}{2(\nu + 2)(\nu + 1)}$$

and let $a := \sqrt{(2\nu + 4)/5} = \sqrt{(n + 2)/5}$. Then, for every $a \le u \le 2a$,

$$Q(u) \ge Q(a) = \frac{2\nu + 4}{5(\nu + 1)} - \frac{(2\nu + 4)^2}{50(\nu + 2)(\nu + 1)} = \frac{8(\nu + 2)}{25(\nu + 1)}.$$

Therefore, when $a \le \pi r\|t\| \le 2a$ we have

$$1 - \frac{2\pi}{\omega_n r^\nu \|t\|^\nu} \, J_\nu(2\pi r\|t\|) \ge \frac{8}{25}.$$

Hence

$$\begin{aligned} &\int_{\mathbb{R}^n} |\mathcal{F}f(t)|^2 \cdot \left| \frac{2\pi}{\omega_n r^\nu \|t\|^\nu} \, J_\nu(2\pi r\|t\|) - 1 \right|^2 dt \\ &\ge \int_{\sqrt{(n+2)/5}\big/\pi r \le \|t\| \le 2\sqrt{(n+2)/5}\big/\pi r} |\mathcal{F}f(t)|^2 \left(\frac{8}{25}\right)^2 dt. \end{aligned}$$

We choose now $r = \pi^{-1}\sqrt{(n+2)/5} \cdot 2^{-j}$ for $j \in \mathbb{N}_0$, $j \geq j_0$. We get

$$\begin{aligned} \frac{n+2}{5\pi^2} 2^{-2j} \int_{\mathbb{R}^n} \psi^2(x)dx &\geq \int_{\mathbb{R}^n} |g(x)|^2 dx \\ &= \int_{\mathbb{R}^n} |\mathcal{F}g(t)|^2 dt \\ &\geq \frac{64}{625} \int_{2^j \leq \|t\| \leq 2^{j+1}} |\mathcal{F}f(t)|^2 dt. \end{aligned}$$

Therefore

$$\int_{2^j \leq \|t\| \leq 2^{j+1}} |\mathcal{F}f(t)|^2 dt \leq c \cdot 2^{-2j} \cdot \|\psi\|_2^2,$$

where the constant $c > 0$ only depends on n. Applying the Cauchy-Schwarz inequality to the functions $|\mathcal{F}f|\chi$ and χ, where χ denotes the indicator function of $\{t \in \mathbb{R}^n : 2^j \leq \|t\| \leq 2^{j+1}\}$, we find

$$\begin{aligned} \int_{2^j \leq \|t\| \leq 2^{j+1}} |\mathcal{F}f(t)|\, dt &\leq \left(\int_{2^j \leq \|t\| \leq 2^{j+1}} |\mathcal{F}f(t)|^2 dt \right)^{1/2} \cdot \left(\int_{2^j \leq \|t\| \leq 2^{j+1}} dt \right)^{1/2} \\ &\leq \left(c \cdot 2^{-2j} \cdot \|\psi\|_2^2 \right)^{1/2} \cdot \left(\omega_n (2^{(j+1)n} - 2^{jn})/n \right)^{1/2} \\ &= \left(c\, \omega_n (2^n - 1)/n \right)^{1/2} \cdot 2^{j(n/2-1)} \cdot \|\psi\|_2. \end{aligned}$$

□

Remark. This lemma is inspired by [4, p.129] and is the n-dimensional version of [3, Lemma 2].

4. Characterizing the support.

Theorem. *Let $\lambda \geq 0$ and $S \in \mathcal{S}'_\lambda(\mathbb{R}^n)$.*

i) *If there exist $k \geq 0$ and U an open set in $\mathbb{R}^n$ on which*

$$\lim_{N \to +\infty} \int_{\|t\| \leq N} (1 - \|t\|^2/N^2)^k\, \mathcal{F}S(t) \mathrm{e}^{2\pi i (x|t)} dt = 0 \tag{4.1}$$

holds uniformly (in x), then S is zero on U.

ii) *Conversely, if $k \geq (n+1)/2 + 2m$, where $m := \lfloor n/4 - 1/2 + \lambda/2 \rfloor + 1$, then (4.1) holds uniformly on any compact subset of $\mathbb{R}^n \setminus \operatorname{supp} S$.*

Proof. To show the first part we consider a bounded open subset W of U and take $\varphi \in A_\lambda(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset W$. Then there exists $f \in L^1(\mathbb{R}^n)$ with $\mathcal{F}f = \varphi$ and moreover $\varphi \in L^1(\mathbb{R}^n)$. Hence $\int_{\mathbb{R}^n} \varphi(x)\mathrm{e}^{2\pi i(x|t)}dx$ is equal to $f(t)$ for almost all $t \in \mathbb{R}^n$ and we get

$$\begin{aligned}
0 &= \lim_{N\to+\infty} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (1 - \|t\|^2/N^2)_+^k \, \mathcal{F}S(t)\mathrm{e}^{2\pi i(x|t)} dt \right) \varphi(x)dx \\
&= \lim_{N\to+\infty} \int_{\mathbb{R}^n} (1 - \|t\|^2/N^2)_+^k \, \mathcal{F}S(t) \left(\int_{\mathbb{R}^n} \mathrm{e}^{2\pi i(x|t)} \varphi(x)dx \right) dt \\
&= \lim_{N\to+\infty} \int_{\mathbb{R}^n} (1 - \|t\|^2/N^2)_+^k \, \mathcal{F}S(t)f(t)dt \\
&= \int_{\mathbb{R}^n} \mathcal{F}S(t)f(t)dt \\
&= \langle S, \varphi\rangle_\lambda,
\end{aligned}$$

where the second equality follows from Fubini and the last but one from Lebesgue dominated convergence theorem. Therefore S is zero on every bounded open set in U, and thus on all U.

To prove the second part we adapt the argument of [3, proof of Theorem 1] and take $x_o \in \mathbb{R}^n \setminus \operatorname{supp} S$ arbitrary and choose $\eta > 0$ such that $d(x_0, \operatorname{supp} S) > 5\eta$. For x in the ball $B(x_0, \eta)$ we have

$$\operatorname{supp}(S \star \delta_{-x}) = \operatorname{supp} S - x \subset \mathbb{R}^n \setminus B(0, 4\eta),$$

where δ_{-x} is the Dirac measure at $-x$: $\delta_{-x}(\varphi) = \varphi(-x)$ if $\varphi \in A(\mathbb{R}^n)$. We take now $\phi \in C^\infty(\mathbb{R}^n)$ radial with $0 \le \phi \le 1$, $\phi = 0$ on $B(0, 2\eta)$ and $\phi = 1$ on $\mathbb{R}^n \setminus B(0, 3\eta)$. We see that, for all $x \in B(x_0, \eta)$,

$$S \star \delta_{-x} = \phi(S \star \delta_{-x}),$$

and therefore

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} (1 - \|t\|^2/N^2)_+^k \, \mathcal{F}S(t)\mathrm{e}^{2\pi i(x|t)} dt \right| \\
&= \left| \int_{\mathbb{R}^n} (1 - \|t\|^2/N^2)_+^k \, \mathcal{F}(S \star \delta_{-x})(t)dt \right| \\
&= |\langle S \star \delta_{-x}, \mathcal{F}\{(1 - \|t\|^2/N^2)_+^k\}\rangle_\lambda|
\end{aligned}$$

$$
\begin{aligned}
&= |\langle S \star \delta_{-x}, {}_kL_N^n \rangle_\lambda| \\
&= |\langle \phi(S \star \delta_{-x}), {}_kL_N^n \rangle_\lambda| \\
&= |\langle S \star \delta_{-x}, \phi \cdot {}_kL_N^n \rangle_\lambda| \\
&\le \|S \star \delta_{-x}\|_{\mathcal{S}'_\lambda} \cdot \|\phi \cdot {}_kL_N^n\|_{A_\lambda} \\
&= \|S\|_{\mathcal{S}'_\lambda} \cdot \|\phi \cdot {}_kL_N^n\|_{A_\lambda},
\end{aligned}
$$

where the last equality follows from $|\mathcal{F}(S \star \delta_{-x})(t)| = |\mathcal{F}S(t)\mathrm{e}^{2\pi i(x|t)}| = |\mathcal{F}S(t)|$. Hence it will suffice to show that

$$\lim_{N\to+\infty} \|\phi \cdot {}_kL_N^n\|_{A_\lambda} = 0.$$

Fix $\varepsilon > 0$. The function $\phi \cdot {}_kL_N^n$ is radial and integrable on $\mathbb{R}^n$; therefore

$$\|\phi \cdot {}_kL_N^n\|_{A_\lambda} = \int_{\mathbb{R}^n} (1+\|t\|)^\lambda |\mathcal{F}(\phi \cdot {}_kL_N^n)(t)|\, dt.$$

Let $j_0 \in \mathbb{N}_0$ such that $\pi^{-1}\sqrt{(n+2)/5} \cdot 2^{-j_0} < \eta$ and write $l := 2m - \lambda$, so that $l > n/2 - 1$. Applying the mean value theorem, we see that the function $\Delta^m(\phi \cdot {}_kL_N^n)$ satisfies the assumptions of the lemma, since $k \ge (n-1)/2 + 2m + 1$. Hence there exists a constant $C > 0$ depending only on n such that, for all $j \in \mathbb{N}$ with $j \ge j_0$,

$$\int_{2^j \le \|t\| \le 2^{j+1}} |\mathcal{F}\{\Delta^m(\phi \cdot {}_kL_N^n)\}(t)|\, dt \le C \cdot 2^{j(n-2)/2} \cdot \|\psi_N\|_2,$$

where $\psi_N(x) := \sup_{\|y-x\|\le\eta} \|\operatorname{grad}\{\Delta^m(\phi \cdot {}_kL_N^n)\}(y)\|$. Moreover, always due to the choice of k, there exists a constant $K > 0$ depending only on n, m, k and ϕ such that $\|\psi_N\|_2 \le K$ for all $N \ge 1$, by (2.4). Then, using the fact that $4\pi^2\|t\|^2\mathcal{F}h(t) = \mathcal{F}(\Delta h)(t)$, we find, for $N \ge 1$,

$$
\begin{aligned}
&\int_{\|t\| \ge 2^{j_0}} (1+\|t\|)^\lambda |\mathcal{F}(\phi \cdot {}_kL_N^n)(t)|\, dt \\
&= \sum_{j=j_0}^{+\infty} \int_{2^j \le \|t\| \le 2^{j+1}} (1+\|t\|)^{2m-l} |\mathcal{F}(\phi \cdot {}_kL_N^n)(t)|\, dt \\
&\le \sum_{j=j_0}^{+\infty} (1+2^j)^{-l} \int_{2^j \le \|t\| \le 2^{j+1}} (2\|t\|)^{2m} |\mathcal{F}(\phi \cdot {}_kL_N^n)(t)|\, dt \\
&\le \sum_{j=j_0}^{+\infty} 2^{-jl} \int_{2^j \le \|t\| \le 2^{j+1}} \pi^{-2m} |\mathcal{F}\{\Delta^m(\phi \cdot {}_kL_N^n)\}(t)|\, dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{j=j_0}^{+\infty} 2^{-jl}\, \pi^{-2m} C\, 2^{j(n-2)/2} \|\psi_N\|_2 \\
&\leq \pi^{-2m}\, C\, K \cdot \sum_{j=j_0}^{+\infty} 2^{(n/2-1-l)j} \\
&< \varepsilon/2,
\end{aligned}
$$

if we choose $j_0 \in \mathbb{N}_0$ big enough so that $\pi^{-1}\sqrt{(n+2)/5} \cdot 2^{-j_0} < \eta$ and $\sum_{j=j_0}^{+\infty} 2^{(n/2-1-l)j} < \varepsilon \pi^{2m}/(CK+1)$ (recall that $l > n/2 - 1$ and that neither C nor K depend on j_0). Moreover, since $k > (n-1)/2$,

$$\lim_{N\to+\infty} \|\mathcal{F}(\phi \cdot {}_kL_N^n)\|_\infty \leq \lim_{N\to+\infty} \|\phi \cdot {}_kL_N^n\|_1 = 0,$$

by (2.3). Hence there exists $N_0 \geq 1$ such that $N \geq N_0$ implies

$$\|\mathcal{F}(\phi \cdot {}_kL_N^n)\|_\infty < \frac{\varepsilon}{2} \cdot \left(\int\limits_{\|t\|\leq 2^{j_0}} (1+\|t\|)^\lambda \, dt \right)^{-1}.$$

Finally we get, for all $N \geq N_0$,

$$
\begin{aligned}
\|\phi \cdot {}_kL_N^n\|_{A_\lambda} &= \int\limits_{\|t\|\leq 2^{j_0}} (1+\|t\|)^\lambda |\mathcal{F}(\phi \cdot {}_kL_N^n)(t)| \, dt \\
&\quad + \int\limits_{\|t\|\geq 2^{j_0}} (1+\|t\|)^\lambda |\mathcal{F}(\phi \cdot {}_kL_N^n)(t)| \, dt \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

We have thus proved that (4.1) holds uniformly on $B(x_0, \eta)$. The conclusion follows. □

Remark. The proof of ii) gives, in fact, more:

Proposition. *Let λ, $\mu \geq 0$, E a closed set in $\mathbb{R}^n$ and W compact in $\mathbb{R}^n \backslash E$. Write $m := \lfloor n/4 - 1/2 + \lambda/2 \rfloor + 1$. If $k \geq (n+1)/2 + 2m$, then (4.1) holds uniformly in $x \in W$ and in $S \in \mathcal{S}'_\lambda(\mathbb{R}^n)$ with $\operatorname{supp} S \subset E$ and $\|S\|_{\mathcal{S}'_\lambda} \leq \mu$.*

Remark. The theorem and the proposition can be stated identically with $(1-\|t\|)_+^k$ (Cesàro means) instead of $(1-\|t\|^2)_+^k$; to see this, use [2], especially lemma 3 p. 292 and the results about $\mathcal{F}\{(1-\|t\|/N)_+^k\}$ at pp. 293–294.

Corollary. *Let* $\sigma \in L^\infty_{loc}(\mathbb{R}^n)$ *be of polynomial growth. There exist for every* $x_0 \in \mathbb{R}^n$ *an integer* $k = k(x_0) \geq 0$ *and a neighbourhood* $U(x_0)$ *of* x_0 *on which*

$$\lim_{N\to+\infty} \int\limits_{\|t\|\leq N} (1 - \|t\|^2/N^2)^k\, \sigma(t)\mathrm{e}^{2\pi i(x|t)} dt = 0 \tag{4.2}$$

holds uniformly if and only if $\sigma = 0$ *almost everywhere.*

Corollary. *Let* $\sigma \in L^\infty_{loc}(\mathbb{R}^n)$ *be of polynomial growth. There exist for every* $x_0 \in \mathbb{R}^n \setminus \{0\}$ *an integer* $k = k(x_0) \geq 0$ *and a neighbourhood* $U(x_0)$ *of* x_0 *on which* (4.2) *holds uniformly if and only if* σ *is equal almost everywhere to a polynomial.*

Proof. Any distribution S with support in $\{0\}$ is of the form

$$\sum_{|\alpha|\leq m} c_\alpha D^\alpha \delta_0$$

with $c_\alpha \in \mathbb{C}$, and $\mathcal{F}[D^\alpha \delta_0](y) = (-2\pi i)^{|\alpha|} \cdot y^\alpha$. □

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